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FRACTIONAL INTEGRAL OPERATORS INVOLVING GENERALIZED STRUVE FUNCTION

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Abstract. Fractional calculus has found many demonstrated applications in extensive fields of engineering and applied science such as fluid mechanics, biological population models, optics, signal processing and control theory.The effectiveness and application of the Struve function in various science problems,here,in this paper we propose to investigate fractional integral operators involving generalized Struve function.The obtained results is general in nature and it is useful to investigate many problems in applied mathematics.

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1. Introduction

The Struve function of order p given by

$$H_p(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+3/2)\,\Gamma\left(k+p+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2k+p+1}, \tag{1}$$

is a particular solution of the non-homogeneous Bessel differential equation [14]

$$x^2 y''(x) + x y'(x) + \left(x^2 - p^2\right) y(x) = \frac{4\left(\frac{x}{2}\right)^{p+1}}{\sqrt{\pi}\,\Gamma(p+1/2)} \tag{2}$$

where Γ is the classical gamma function. The generalized Struve function given by Bhowmick [1]

$$H_l^{\lambda}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{t}{2}\right)^{2k+l+1}}{\Gamma\left(\lambda k + l + \frac{3}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}, \lambda > 0. \tag{3}$$

Another generalization is given by Kant [4]

$$H_l^{\lambda,\alpha}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+l+1}}{\Gamma\left(\lambda k + l + \frac{3}{2}\right)\Gamma\left(\alpha k+\frac{3}{2}\right)}, \lambda > 0, \alpha > 0. \tag{4}$$

The generalized Struve function of four parameters given in[9]:

$$H_{l,\xi}^{\lambda}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+l+1}}{\Gamma\left(\lambda k + \frac{l}{\xi} + \frac{3}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}, \lambda > 0, \xi > 0 \tag{5}$$

and [10] gave more generalized Struve function which is defined by

$$(6)\qquad H_{p,\mu}^{\lambda,\alpha}(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k+\mu)\Gamma\left(\lambda k+p+\frac{3}{2}\right)} \left(\tfrac{x}{2}\right)^{2k+p+1}, p\in\mathbb{C},$$

where $\lambda>0, \alpha>0$ and μ is an arbitrary parameter.

Now, we give a more generalized form of Struve function as:

$$(7)\qquad H_{p,\mu,b,c}^{\lambda,\alpha}(x) := \sum_{k=0}^{\infty} \frac{(-c)^k}{\Gamma(\alpha k+\mu)\Gamma\left(\lambda k+p+\frac{b+2}{2}\right)} \left(\tfrac{x}{2}\right)^{2k+p+1}, p,b,c\in\mathbb{C},$$

where $\lambda>0, \alpha>0$ and μ is an arbitrary parameter. The Struve function and its more generalizations are found in many papers, interesting readers can refer ([1]- [13]). The generalized Wright hypergeometric function ${}_p\psi_q(z)$ is given by the series

$$(8)\qquad {}_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\middle|\, z\right] = \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}\frac{z^k}{k!},$$

where $a_i, b_j\in\mathbb{C}$, and real $\alpha_i,\beta_j\in\mathbb{R}$ $(i=1,2,\dots,p; j=1,2,\dots,q)$. Asymptotic behavior of this function for large values of argument of $z\in\mathbb{C}$ were studied in [3] and under the condition

$$(9)\qquad \sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i>-1$$

was found in the work of ([15],[16]). Properties of this generalized Wright function were investigated in [5] (see also [6, 7]. In particular, it was proved [5] that ${}_p\psi_q(z)$, $z\in\mathbb{C}$ is an entire function under the condition (9).

The generalized hypergeometric function represented as follows [8]:

$$(10)\qquad {}_pF_q\left[\begin{array}{c}(\alpha_p);\\(\beta_q);\end{array} z\right] = \sum_{n=0}^{\infty}\frac{\Pi_{j=1}^{p}(\alpha_j)_n}{\Pi_{j=1}^{p}(\beta_j)_n}\frac{z^n}{n!},$$

provided $p\le q; p=q+1$ and $|z|<1$

where $(\lambda)_n$ is well known Pochhammer symbol defined for (for $\lambda\in\mathbb{C}$) (see [8])

$$(11)\quad (\lambda)_n := \begin{cases} 1 & (n=0)\\ \lambda(\lambda+1)\dots(\lambda+n-1) & (n\in N:=\{1,2,3\dots\})\end{cases}$$

$$(12)\qquad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \qquad \left(\lambda\in C\backslash Z_0^-\right).$$

where Z_0^- is the set of non positive integers.

If we put $\alpha_1=\dots=\alpha_p=\beta_1=\dots=\beta_q$ in (8),then (10) is a special case of the generalized Wright function:

(13)

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{c}(\alpha_1,1),\dots,(\alpha_p,1);\\(\beta_1,1),\dots,(\beta_q,1);\end{array} z\right] = \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left[\begin{array}{c}\alpha_1,\dots,\alpha_p;\\\beta_1,\dots,\beta_q;\end{array} z\right]$$

In this paper, we investigated the fractional integral operators of generalized Struve function given in (7). For this purpose, we recall the following lemmas.

Lemma 1.1. [6] *Let* $\alpha, \beta, \eta \in \mathbb{C}$, $\Re(\alpha) > 0$ *and* $\lambda > \max[0, \beta - \eta]$, *then the following relation holds:*

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)\Gamma(\lambda+\eta-\beta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\alpha+\eta)} x^{\lambda-\beta-1} \tag{14}$$

Lemma 1.2. [6] *Let* $\alpha, \beta, \eta \in \mathbb{C}$, $\Re(\alpha) > 0$ *and* $\lambda > \max[0, \beta - \eta]$, *then the following relation holds:*

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\lambda-1}\right)(x) = \frac{\Gamma(\eta-\lambda+1)\Gamma(\beta-\lambda+1)}{\Gamma(1-\lambda)\Gamma(\alpha+\beta+\eta-\lambda+1)} x^{\lambda-\beta-1} \tag{15}$$

2. Main Results

We devote this section to the derive the fractional integral representation of (6) The following theorem is first presented.

Theorem 2.1. *Let* $\lambda, \sigma, \nu, \alpha, \beta, \mu \in \mathbb{C}$, $\Re(p) > -1$ *and* $\rho > \max[0, \sigma - \nu]$. *Then there holds the formula:*

$$\begin{aligned}&\left(I_{0,+}^{\lambda,\sigma,\nu} t^{\rho-1} H_{p,\mu,b,c}^{\beta,\alpha}(t)\right)(x) = \frac{x^{\rho+p-\sigma}}{2^{p+1}} \\ (16)\quad &\times \quad {}_3\Psi_4\left[\begin{array}{c}(\rho+p+1,2),(\rho+p+1+\nu-\sigma,2),(1,1);\\ (\mu,\alpha),(p+\frac{b+2}{2},\beta),(\rho+p+1-\sigma,2),(\rho+p+1+\lambda+\nu,2)\end{array}, -\frac{cx^2}{4}\right].\end{aligned}$$

Proof. Let the left hand side of (16) denoted by $\mathfrak{I}_1$ and using the definition of generalized Struve function given in (7), we get

$$\begin{aligned}\mathfrak{I}_1 &= \left(I_{0,+}^{\lambda,\sigma,\nu} t^{\rho-1} H_{p,\mu,b,c}^{\beta,\alpha}(t)\right)(x)\\ &= \left(I_{0,+}^{\lambda,\sigma,\nu} t^{\rho-1} \sum_{r=0}^{\infty} \frac{(-c^r)}{\Gamma(\alpha r+\mu)\Gamma(\beta r+p+\frac{b+2}{2})} \left(\tfrac{t}{2}\right)^{2r+p+1}\right)(x),\end{aligned}$$

Interchanging the integration and summation, gives

$$\mathfrak{I}_1 = \sum_{r=0}^{\infty} \frac{(-c^r)\left(\frac{1}{2}\right)^{2r+p+1}}{\Gamma(\alpha r+\mu)\Gamma(\beta r+p+\frac{b+2}{2})} \left(I_{0,+}^{\lambda,\sigma,\nu} t^{(\rho+p+1+2r)-1}\right)(x).$$

Applying Lemma 1, we get

$$\begin{aligned}\mathfrak{I}_1 &= \sum_{r=0}^{\infty} \frac{(-c^r)\left(\frac{1}{2}\right)^{2r+p+1}}{\Gamma(\alpha r+\mu)\Gamma(\beta r+p+\frac{b+2}{2})}\\ &\times \frac{\Gamma(\rho+p+1+2r)\Gamma(\rho+p+1+2r+\nu-\sigma)}{\Gamma(\rho+p+1+2r-\sigma)\Gamma(\rho+p+1+2r+\lambda+\nu)},\\ &= \frac{x^{\rho+p-\sigma}}{2^{p+1}} \sum_{r=0}^{\infty} \frac{(-c^r)\left(\frac{1}{2}\right)^{2r+p+1}}{\Gamma(\alpha r+\mu)\Gamma(\beta r+p+\frac{b+2}{2})}\\ &\times \frac{\Gamma(\rho+p+1+2r)\Gamma(\rho+p+1+2r+\nu-\sigma)(-1)^r}{\Gamma(\alpha r+\mu)\Gamma(\beta r+p+\frac{b+2}{2})\Gamma(\rho+p+1+2r-\sigma)\Gamma(\rho+p+1+2r+\lambda+\nu)}\end{aligned}$$

In view of definition of generalized Wright function given in (13), we arrived the desired result. □

Corollary 2.2. *If we take $b = c = 1$ in theorem 2.1, we get the fractional integration of Struve function in four parameters given in* [10] *as*

$$\left(I_{0,+}^{\lambda,\sigma,\nu} t^{\rho-1} H_{p,\mu,1,1}^{\beta,\alpha}(t)\right)(x) = \frac{x^{\rho+p-\sigma}}{2^{p+1}} \tag{17}$$
$$\times \quad {}_3\Psi_4\left[\begin{matrix} (\rho+p+1,2), (\rho+p+1+\nu-\sigma,2), (1,1); \\ (\mu,\alpha), (p+\frac{3}{2},\beta), (\rho+p+1-\sigma,2), (\rho+p+1+\lambda+\nu,2) \end{matrix}, -\frac{x^2}{4}\right].$$

Corollary 2.3. *If we take $b = c = 1$ and $\mu = \frac{3}{2}$ in theorem 2.1, we get the fractional integration of Struve function given in* [4] *as*

$$\left(I_{0,+}^{\lambda,\sigma,\nu} t^{\rho-1} H_{p,\frac{3}{2},1,1}^{\beta,\alpha}(t)\right)(x) = \frac{x^{\rho+p-\sigma}}{2^{p+1}} \tag{18}$$
$$\times \quad {}_3\Psi_4\left[\begin{matrix} (\rho+p+1,2), (\rho+p+1+\nu-\sigma,2), (1,1); \\ (1,\alpha), (p+\frac{3}{2},\beta), (\rho+p+1-\sigma,2), (\rho+p+1+\lambda+\nu,2) \end{matrix}, -\frac{x^2}{4}\right].$$

Corollary 2.4. *If we set $b = c = \alpha = 1$ and $\mu = \frac{3}{2}$ in theorem 2.1, we get the fractional integration of generalized Struve function* [1] *as*

$$\left(I_{0,+}^{\lambda,\sigma,\nu} t^{\rho-1} H_{p,\frac{3}{2},1,1}^{\beta,1}(t)\right)(x) = \frac{x^{\rho+p-\sigma}}{2^{p+1}} \tag{19}$$
$$\times \quad {}_3\Psi_4\left[\begin{matrix} (\rho+p+1,2), (\rho+p+1+\nu-\sigma,2), (1,1); \\ (1,1), (p+\frac{3}{2},\beta), (\rho+p+1-\sigma,2), (\rho+p+1+\lambda+\nu,2) \end{matrix}, -\frac{x^2}{4}\right].$$

Corollary 2.5. *If we set $b = c = 1$, $\alpha = \beta = 1$ and $\mu = \frac{3}{2}$ in theorem 2.1, we get the fractional integration of Struve function:*

$$\left(I_{0,+}^{\lambda,\sigma,\nu} t^{\rho-1} H_{p,\frac{3}{2},1,1}^{1,1}(t)\right)(x) = \frac{x^{\rho+p-\sigma}}{2^{p+1}} \tag{20}$$
$$\times \quad {}_3\Psi_4\left[\begin{matrix} (\rho+p+1,2), (\rho+p+1+\nu-\sigma,2), (1,1); \\ (1,1), (p+\frac{3}{2},1), (\rho+p+1-\sigma,2), (\rho+p+1+\lambda+\nu,2) \end{matrix}, -\frac{x^2}{4}\right].$$

Theorem 2.6. *Let $\lambda,\sigma,\nu,\alpha,\beta,\mu \in \mathbb{C}, \Re(p) > -1, \Re(\lambda) > 0$ and $\Re(\rho) < 1 + min[\Re(\sigma), \Re(\nu)]$. Then there holds the formula:*

$$\left(I_{-}^{\lambda,\sigma,\nu} t^{\rho-1} H_{p,\mu,b,c}^{\beta,\alpha}(1/t)\right)(x) = \frac{x^{\rho-p-\sigma-2}}{2^{p+1}} \tag{21}$$
$$\times\, {}_3\Psi_4\left[\begin{matrix} (\sigma-\rho+p+2,2), (\nu-\rho+p+2,2), (1,1); \\ (\mu,\alpha), (p+\frac{b+2}{2},\beta), (2-\rho+p,2), (\lambda+\sigma+\nu-\rho+2,2) \end{matrix}, -\frac{c}{4x^2}\right].$$

Proof. Let the left hand side of (21) denoted by $\mathfrak{I}_2$ and using the definition of generalized Struve function given in (7), we get

$$\mathfrak{I}_2 = \left(I_{-}^{\lambda,\sigma,\nu} t^{\rho-1} H_{p,\mu,b,c}^{\beta,\alpha}(1/t)\right)(x)$$
$$= \left(I_{-}^{\lambda,\sigma,\nu} t^{\rho-1} \sum_{r=0}^{\infty} \frac{(-c^r)}{\Gamma(\alpha r+\mu)\Gamma(\beta r+p+\frac{b+2}{2})} \left(\frac{1}{2t}\right)^{2r+p+1}\right)(x),$$

Interchanging the integration and summation, gives

$$\mathfrak{I}_2 = \sum_{r=0}^{\infty} \frac{(-c^r)\left(\frac{1}{2}\right)^{2r+p+1}}{\Gamma(\alpha r+\mu)\Gamma(\beta r+p+\frac{b+2}{2})} \left(I_{-}^{\lambda,\sigma,\nu} t^{(\rho-p-1-2r)-1}\right)(x).$$

Applying Lemma 2, we get

$$\mathfrak{I}_2 = \sum_{r=0}^{\infty} \frac{(-c^r)\left(\frac{1}{2}\right)^{2r+p+1}}{\Gamma(\alpha r+\mu)\Gamma(\beta r+p+\frac{b+2}{2})}$$
$$\times \frac{\Gamma(\sigma-\rho+p+2+2r)\Gamma(\nu-\rho+p+2+2r)}{\Gamma(1-\rho+p+1+2r)\Gamma(\lambda+\sigma+\nu-\rho+p+2+2r)} x^{\rho-p-2-2r-\sigma},$$
$$= \frac{x^{\rho-p-2-\sigma}}{2^{p+1}} \sum_{r=0}^{\infty} \frac{(-c^r)\left(\frac{1}{2}\right)^{2r+p+1}}{\Gamma(\alpha r+\mu)\Gamma(\beta r+p+\frac{b+2}{2})}$$
$$\times \frac{\Gamma(\sigma-\rho+p+2+2r)\Gamma(\nu-\rho+p+2+2r)}{\Gamma(1-\rho+p+1+2r)\Gamma(\lambda+\sigma+\nu-\rho+p+2+2r)} x^{\rho-p-2-2r-\sigma}$$

In view of definition (13), we arrived the desired result. □

Corollary 2.7. *If we take $b = c = 1$ in theorem 2.6, we get the fractional integration of Struve function in four parameters given in* [10] *as*

$$\left(I_{-}^{\lambda,\sigma,\nu} t^{\rho-1} H_{p,\mu,1,1}^{\beta,\alpha}(1/t)\right)(x) = \frac{x^{\rho-p-\sigma-2}}{2^{p+1}}$$

(22)

$$\times\, {}_3\Psi_4 \left[\begin{matrix} (\sigma-\rho+p+2,2),(\nu-\rho+p+2,2),(1,1); \\ (\mu,\alpha),(p+\frac{3}{2},\beta),(2-\rho+p,2),(\lambda+\sigma+\nu-\rho+2,2) \end{matrix}, -\frac{1}{4x^2}\right].$$

Corollary 2.8. *If we set $b = c = 1$ and $\mu = \frac{3}{2}$ in theorem 2.6, we get the fractional integration of Struve function given in* [4] *as*

$$\left(I_{-}^{\lambda,\sigma,\nu} t^{\rho-1} H_{p,\frac{3}{2},1,c}^{\beta,\alpha}(1/t)\right)(x) = \frac{x^{\rho-p-\sigma-2}}{2^{p+1}}$$

(23)

$$\times\, {}_3\Psi_4 \left[\begin{matrix} (\sigma-\rho+p+2,2),(\nu-\rho+p+2,2),(1,1); \\ (1,\alpha),(p+\frac{3}{2},\beta),(2-\rho+p,2),(\lambda+\sigma+\nu-\rho+2,2) \end{matrix}, -\frac{1}{4x^2}\right].$$

Corollary 2.9. *If we take $b = c = \alpha = 1$ and $\mu = \frac{3}{2}$ in theorem 2.6, we get the fractional integration of generalized Struve function* [1] *as*

$$\left(I_{-}^{\lambda,\sigma,\nu} t^{\rho-1} H^{\beta,1}_{p,\frac{3}{2},1,1}(1/t)\right)(x) = \frac{x^{\rho-p-\sigma-2}}{2^{p+1}} \tag{24}$$
$$\times\, {}_3\Psi_4\left[\begin{array}{c}(\sigma-\rho+p+2,2),(\nu-\rho+p+2,2),(1,1);\\(1,1),(p+\frac{3}{2},\beta),(2-\rho+p,2),(\lambda+\sigma+\nu-\rho+2,2)\end{array}, -\frac{1}{4x^2}\right].$$

Corollary 2.10. *IIf we set $b = c = 1$, $\alpha = \beta = 1$ and $\mu = \frac{3}{2}$ in theorem 2.6, we get the fractional integration of Struve function:*

$$\left(I_{-}^{\lambda,\sigma,\nu} t^{\rho-1} H^{1,1}_{p,\frac{3}{2},1,1}(1/t)\right)(x) = \frac{x^{\rho-p-\sigma-2}}{2^{p+1}} \tag{25}$$
$$\times\, {}_3\Psi_4\left[\begin{array}{c}(\sigma-\rho+p+2,2),(\nu-\rho+p+2,2),(1,1);\\(1,1),(p+\frac{3}{2},1),(2-\rho+p,2),(\lambda+\sigma+\nu-\rho+2,2)\end{array}, -\frac{1}{4x^2}\right].$$

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